

# THE RELATIVISTIC DIRAC-MORSE PROBLEM VIA SUSY QM

R. de Lima Rodrigues<sup>1</sup>

Centro Brasileiro de Pesquisas Físicas (CBPF)

Rua Dr. Xavier Sigaud, 150, CEP 22290-180, Rio de Janeiro-RJ, Brazil

A. N. Vaidya

Instituto de Física - Universidade Federal do Rio de Janeiro

Caixa Postal 68528 - CEP 21945-970, Rio de Janeiro, Brazil

## Abstract

The Morse problem is investigated in relativistic quantum mechanics.

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## 1 Introduction

The Dirac oscillator was first formulated and investigated in the context of nonrelativistic quantum mechanics by Moshinsky-Szczepaniak [1]. They construct a Dirac Hamiltonian, linear in the momentum  $\vec{p}$  and position  $\vec{r}$ , whose square leads to the ordinary harmonic oscillator in the nonrelativistic limit. In 1992, Dixit *et al.* instead of modifying the momentum, considered a Dirac oscillator with scalar coupling, by modification of the mass term [2].

In a recent paper Alhaidari has proposed a generalization for Moshinsky's equation which contains the oscillator and the Coulomb problem as special cases. He also claims to have formulated and solved the Dirac-Morse problem and obtaining its relativistic bound states and spinor wave functions [3]. In a later paper a similar method is applied to include shape invariant potentials via supersymmetry in quantum mechanics (SUSYQM) [4]. However, his calculation contains serious mistakes as we explain below [5]. We also explain another way of treating the problem.

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<sup>1</sup>Permanent address: Departamento de Ciências Exatas e da Natureza, Universidade Federal de Campina Grande, Cajazeiras - PB, 58.900-000-Brazil. E-mail to RLR is [rafaelr@cbpf.br](mailto:rafaelr@cbpf.br) or [rafael@fisica.ufpb.br](mailto:rafael@fisica.ufpb.br), the e-mail to ANV is [vaidya@if.ufrj.br](mailto:vaidya@if.ufrj.br).

## 2 Problem with Alhaidari's Calculation

The Hamiltonian that appears in the work of Alhaidari [3] does not lead to his equation (1). The Hamiltonian should be in the notation is that of Bjorken and Drell [6].

$$H = \boldsymbol{\alpha} \cdot (\mathbf{p} - i\beta \hat{\mathbf{r}} W(r)) + \beta M + V(r) \quad (1)$$

where  $\hat{\mathbf{r}} = \frac{\mathbf{r}}{r}$  and we consider  $\hbar = 1$  and the mass as being  $M$ . Due to the matrix  $\beta$  accompanying  $W$  in the Hamiltonian Alhaidari's interpretation of the vector  $(V, \hat{\mathbf{r}}W)$  as an electromagnetic potential is incorrect. To verify our assertion we put

$$\Psi = \begin{pmatrix} \frac{iG_{\ell j}}{r} \phi_{jm}^{\ell} \\ \frac{F_{\ell j}}{r} \boldsymbol{\sigma} \cdot \hat{\mathbf{r}} \phi_{jm}^{\ell} \end{pmatrix} \quad (2)$$

where  $\phi_{jm}^{\ell}$  are as defined in reference [6]. Then using the relations

$$\boldsymbol{\sigma} \cdot \mathbf{p} \frac{f(r)}{r} \phi_{jm}^{\ell} = -\frac{i}{r} \left( \frac{df}{dr} + \frac{\kappa f}{r} \right) \boldsymbol{\sigma} \cdot \hat{\mathbf{r}} \phi_{jm}^{\ell} \quad (3)$$

and

$$\boldsymbol{\sigma} \cdot \mathbf{p} \boldsymbol{\sigma} \cdot \hat{\mathbf{r}} \frac{f(r)}{r} \phi_{jm}^{\ell} = -\frac{i}{r} \left( \frac{df}{dr} - \frac{\kappa f}{r} \right) \phi_{jm}^{\ell} \quad (4)$$

where  $\kappa = \pm(j + \frac{1}{2})$  for  $\ell = j \pm \frac{1}{2}$ , and defining

$$\Phi = \begin{pmatrix} G_{\ell j} \\ F_{\ell j} \end{pmatrix} \quad (5)$$

we get the radial equations

$$\left( -i\rho_2 \frac{d}{dr} + \rho_1 \left( W + \frac{\kappa}{r} \right) - (E - V) + M\rho_3 \right) \Phi = 0 \quad (6)$$

where  $\rho_i$  are the Pauli matrices. The last equation corresponds to Alhaidari's equation (1) where the quantum numbers  $l$  and  $j$  are omitted incorrectly.

Next, there is no reason for the functions  $V(r)$  and  $W(r)$  which appear in the Hamiltonian to depend on the angular quantum numbers which make their appearance only when we separate variables to solve the Dirac equation. Hence his choice of the constraint

$$W(r) = \frac{\alpha}{S} V(r) - \frac{\kappa}{r} \quad (7)$$

cannot be satisfied since  $\kappa = \pm(j + \frac{1}{2})$  for  $l = j \pm \frac{1}{2}$  is not a fixed quantity. Since the Dirac operator

$$K = \gamma^0(1 + \boldsymbol{\Sigma} \cdot \mathbf{L}) \quad (8)$$

satisfies the property

$$K\phi_{jm}^\ell = -\kappa\phi_{jm}^\ell. \quad (9)$$

Alhaidari could have avoided the contradiction by taking the Hamiltonian to be

$$H = \boldsymbol{\alpha} \cdot (\mathbf{p} - i\beta\hat{\mathbf{r}}(W(r) + \frac{K}{r})) + \beta M + V(r) \quad (10)$$

which leads to the radial equations

$$\left(-i\rho_2\frac{d}{dr} + W\rho_1 - (E - V) + M\rho_3\right)\Phi = 0. \quad (11)$$

Applying the transformation

$$\Phi = e^{-i\rho_2\eta}\hat{\Phi} \quad (12)$$

we have

$$\begin{aligned} &\left[-i\rho_2\frac{d}{dr} - (E - V) + \rho_1(W\cos 2\eta - M\sin 2\eta)\right]\hat{\Phi} \\ &+ [\rho_3(W\sin 2\eta + M\cos 2\eta)]\hat{\Phi} = 0. \end{aligned} \quad (13)$$

Choosing

$$W = \frac{V}{\sin 2\eta}. \quad (14)$$

Hence

$$\begin{aligned} &\left[-i\rho_2\frac{d}{dr} - E + V(1 + \rho_3) + \rho_1\left(\frac{V}{\tan 2\eta} - M\sin 2\eta\right)\right]\hat{\Phi} \\ &+ [\rho_3M\cos 2\eta]\hat{\Phi} = 0 \end{aligned} \quad (15)$$

which gives

$$\hat{F}_{\ell j} = \frac{1}{E + M\cos 2\eta}\left(\frac{d}{dr} + \frac{V}{\tan 2\eta} - M\sin 2\eta\right)\hat{G}_{\ell j} \quad (16)$$

and

$$\left[-\frac{d^2}{dr^2} + \left(\frac{V}{\tan 2\eta}\right)^2 + 2EV - \frac{1}{\tan 2\eta}\frac{dV}{dr} - (E^2 - M^2)\right]\hat{G}_{\ell j} = 0. \quad (17)$$

The last three equations correspond to equations (3-5) of Alhaidari. The choice  $W = \frac{V}{\sin 2\eta}$  gives equations (4) and (7) of Alhaidari.

The resulting energy levels will be degenerate in  $l, j, m$  in contrast to what happens in the case of relativistic Coulomb and relativistic harmonic oscillator problems.

### 3 The Morse potential: An alternative treatment

Even if one interprets the results of Alhaidari as corresponding only to  $\ell = 0, s = 0, k = -1$ , the unitary transformation is puzzling. It does not appear at all if we start with  $V = 0$  as in the treatment of Castaños *et al.* earlier [7]. Then equation (16) gives  $F$  in terms of  $G$  and the second order equation for  $k = -1$  is

$$\left[ -\frac{d^2}{dr^2} + \left(W - \frac{1}{r}\right)^2 + \frac{d}{dr}\left(W - \frac{1}{r}\right) - (E^2 - M^2) \right] \hat{G}_{\ell j} = 0. \quad (18)$$

The choice  $W = \frac{1}{r} + A - Be^{-\lambda r}$  gives the S-wave Morse potential in quantum mechanics with an additional  $A^2$  term in the potential. Although the relativistic spectrum is different, the same non-relativistic limit is obtained.

### 4 Conclusion

In this communication we have pointed out the limitation of a published treatment of the relativistic Morse potential problem. We also present an alternative treatment which is similar to the supersymmetric formulation of the Dirac oscillator as given by Castaños *et al.* [7]. A similar modification holds for relativistic shape invariant potentials as well.

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